MATH 2020 A
Fall 2020

Lecture 24

- differential forms
- quick review.

Last time we showed that how the general Stokes' thm

$$\int_M d\omega = \int_{\partial M} \omega$$

gives a unified treatment of Green's, Stokes' and divergence theorem. We continue to explore this connection.

A differential form ω is called closed if $d\omega = 0$, and it is called exact if $\omega = d\eta$ for some η. Thm 2 in Lecture 23 tells us that every exact form is closed. We have the following converse.

Theorem 1 Let ω be a differential form in a convex set Ω. Then it is closed iff it is exact.

This is Poincaré Lemma. (Many pfs are available in the internet.)

Theorem 2 For any function Φ,

(a) $\nabla \times \nabla \Phi = \vec{0}$. (a gradient v.f has no curl)

For any v.f. $\vec{F}$,

(b) $\nabla \cdot \nabla \times \vec{F} = 0$ (a curl v.f. has no div.)

The proof is by direct computations. ~~We now express they in differential forms.~~

For the 1-form $\eta = f\,dx + g\,dy + h\,dz$,

$$d\eta = (g_x - f_y)\,dx\wedge dy + (f_z - h_x)\,dz\wedge dx + (h_y - g_z)\,dy\wedge dz.$$

If we let $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$, then

$$\nabla\times\vec{F} = (h_y - g_z)\hat{i} + (f_z - h_x)\hat{j} + (g_x - f_y)\hat{k}$$

Therefore, $\nabla\cdot\vec{A} = 0$ iff $d\omega = 0$, and

$$\vec{A} = \nabla\times\vec{F} \quad \text{iff} \quad \omega = d\eta.$$

Question (B) is the same as; when ω is closed, is it exact?

Let's point out Helmholtz deomposition: for any v.f. $\vec{F}$,

there are fcn Φ and v.f. $\vec{A}$ s.t

$$\vec{F} = \underbrace{\nabla\Phi}_{\text{curl free}} + \underbrace{\nabla\times\vec{A}}_{\text{div. free}}.$$

Finally, suppose a gradient v.f. $\nabla\Phi$ is divergence free in Ω (that's no gain no loss in Ω). We have

$$\nabla\cdot\nabla\Phi = 0, \quad \text{ie}$$

$$\nabla\cdot\left(\frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}\right) = 0, \quad \text{or}$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

Set $\Delta \Phi = \nabla^2 \Phi$

$$= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0.$$

The equation

$$\Delta \Phi = 0$$

is called the Laplacian equation — Probably the most important partial differential equation.

Any solution of the Laplace equation is called a harmonic function.